

Effective Fluid FLRW Cosmologies of Minimal Massive Gravity

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Abstract

By using a solution ansatz we partially decouple the metric and the Stückelberg sectors of the minimal massive gravity (MMGR). In this scheme for a diagonal physical metric we find the general solutions for the scalars of the theory and the particular fiducial (background) metric which leads to these solutions. Then we adopt this general formalism to construct the derivation of new Friedmann-Lemaitre-Robertson-Walker (FLRW) cosmologies of the theory in the presence of a so-called effective ideal fluid which arises from our solution ansatz as a modifying, non-physical source for the Einstein and the corresponding Friedmann equations.

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1 Introduction

The de Rham, Gabadadze, Tolley (dRGT) massive gravity [1, 2] which is Boulware-Deser (BD) [3, 4] ghost-free and a non-linear continuation of the Fierz-Pauli [5] massive gravity theory is reformulated in a series of works [6, 7, 8] to accommodate a general background or fiducial metric in a minimal or a more general formalism. These theories which include the dRGT theory as a special case are also proven to be ghost-free at all orders. Both for the minimal and the general cases cosmological solutions of these theories have been intensively studied in the recent literature. The reader may refer to a fair review of these works as well as their achievements and shortcomings in [9]. In particular, it has been shown that for flat background metric choice in spite of the existence of spatially open FLRW solutions [10, 11] there exists no spatially flat or closed FLRW solutions [12]. On the other hand there exist homogeneous and isotropic solutions for the de Sitter [13] and the FLRW type [14] background or fiducial metrics.

In the following work we focus on the minimal massive gravity [6]. We propose an ansatz which contributes an arbitrary effective energy-momentum tensor to the metric sector which admits Einstein equations modified by the existence of this effective non-physical matter originated source. In this manner the metric sector contains neither the Stückelberg scalars nor the fiducial metric explicitly. We later prove that when one considers a general diagonal physical metric solution of the corresponding Einstein equations the Stückelberg sector can be exactly solved for a functionally parametrized set of background metrics. In doing this unlike the common approach of pre-determining the background metric in the above mentioned literature we deduce it from the solution ansatz as a function of the physical metric solved via the Einstein equations and the arbitrarily chosen effective energy-momentum source (which is arbitrary up to a continuity equation). We determine the ansatz satisfying fiducial metrics up to a family of integrable functions which lead also to the solutions of the Stückelberg scalars. Within this formalism as a physical example we present the solutions of the background metric and the Stückelberg scalars as a function of the scale factor and the arbitrarily fixed effective energy-momentum tensor for the homogeneous and the isotropic FLRW physical metric. We also derive the corresponding Friedmann and the acceleration equations which only differ from the GR-originating ones by the presence of an effective ideal fluid arising from the solution ansatz we proposed which also can be considered as a form of gravitational matter

rather than a physical one.

In Section two we discuss the details of our solution mechanism. Section three contains the derivation of the solutions of the Stückelberg scalars and the background fiducial metric for a diagonal choice of the physical metric. Finally, in Section four we adopt the results of Section three to construct the FLRW cosmological solutions of the minimal massive gravity within the formalism introduced in Section two.

2 The set-up

The minimal ghost-free massive gravity action coupled to matter via [6] is

$$S_{MMGR+MATT} = -M_p^2 \int \left[R * 1 + 2m^2 \text{tr}(\sqrt{\Sigma}) * 1 + \Lambda' * 1 \right] + S_{MATT}, \quad (2.1)$$

with M_p being the planck mass, m is the graviton mass, R is the Ricci scalar, and the $*$ is the Hodge star operator. In the above action $\Lambda' = \Lambda - 6m^2$. The four by four matrix Σ is

$$(\Sigma)_\nu^\mu = g^{\mu\rho} \partial_\rho \phi^a \partial_\nu \phi^b \bar{f}_{ab}(\phi^c). \quad (2.2)$$

Here $g^{\mu\nu}$ is the inverse metric. ϕ^a are the Stückelberg scalar fields and $\bar{f}_{ab}(\phi^c)$ is the fiducial background metric yet not specified and which can arbitrarily be chosen in (2.1) in a relative physical context to generate appropriate physical solutions. The spacetime indices $\mu, \nu \dots$ as well as the Stückelberg indices $a, b, c \dots$ run on $0, 1, 2, 3$. We have also introduced the square root matrix obeying $\sqrt{\Sigma} \sqrt{\Sigma} = \Sigma$

Now, referring to [6] we can write the metric equation corresponding to (2.1) as

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \frac{1}{2} \Lambda g_{\mu\nu} + \frac{1}{2} m^2 [g \sqrt{\Sigma} + \sqrt{\Sigma}^T g]_{\mu\nu} + m^2 g_{\mu\nu} (3 - \text{tr}[\sqrt{\Sigma}]) = G_N T_{\mu\nu}, \quad (2.3)$$

where $T_{\mu\nu}$ is the physical energy-momentum contribution of the matter term in (2.1). The Stückelberg scalar field equations can also equivalently be achieved from the covariant derivative-constancy of the metric equation as in [6], they read

$$\nabla_\mu ([\sqrt{\Sigma}]_\nu^\mu + [g^{-1} \sqrt{\Sigma}^T g]_\nu^\mu - 2 \text{tr}[\sqrt{\Sigma}] \delta_\nu^\mu) = 0, \quad (2.4)$$

here ∇_μ is the covariant derivative corresponding to the Levi-Civita connection of the metric g . It acts on $\sqrt{\Sigma}$ as a (1,1) tensor. Following an extended but a similar scheme like in [15] we will decouple the Einsteinian gravity sector from the massive one by introducing a solution ansatz of the form

$$\frac{1}{2}[g\sqrt{\Sigma} + \sqrt{\Sigma}^T g] - \text{tr}[\sqrt{\Sigma}]g = C_1 g + C_2 \tilde{T}, \quad (2.5)$$

where C_1 is a dimensionless, and C_2 is a dimensionful arbitrary constants which can be used to physically tune our solution model to various forms. (2.5) is written as a matrix relation and \tilde{T} is a four by four arbitrary (at this stage) matrix functional which will play the role of effective matter in the Einstein or the metric sector which we will call dynamical. To appreciate our terminology of dynamical-kinematical decoupling between the physical metric g and the Stückelberg scalars and the fiducial metric it is enough to substitute (2.5) in (2.3) which yields

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \tilde{\Lambda}g_{\mu\nu} = G_N T_{\mu\nu} - C_2 m^2 \tilde{T}_{\mu\nu}, \quad (2.6)$$

where we define $\tilde{T}_{\mu\nu} \equiv [\tilde{T}]^\mu{}_\nu$. We have the effective cosmological constant

$$\tilde{\Lambda} = \frac{1}{2}\Lambda - 3m^2 - C_1 m^2. \quad (2.7)$$

Now, the effective contribution of the ansatz (2.5) to the metric sector is more explicit in (2.6) as these equations are the usual Einstein equations for the metric g which have a yet arbitrary effective energy-momentum tensor addition on the right hand side. We have called \tilde{T} an energy-momentum tensor because if we want the solutions of our ansatz (2.5) to be also the solutions of the scalar field equations (2.4) then \tilde{T} must satisfy the constraint

$$\nabla^\mu \tilde{T}_{\mu\nu} = 0. \quad (2.8)$$

This can easily be seen from (2.5) by applying a covariant derivative on both sides

$$\nabla^\mu \left[\frac{1}{2}[g\sqrt{\Sigma} + \sqrt{\Sigma}^T g] - \text{tr}[\sqrt{\Sigma}]g \right]_{\mu\nu} = \nabla^\mu (C_1 g_{\mu\nu} + C_2 \tilde{T}_{\mu\nu}), \quad (2.9)$$

here if one uses the metric compatibility and imposes the constraint (2.8) on the right hand side then the left hand side becomes zero. This result leads

us to the scalar field equations (2.4) upon using the metric compatibility, and index lowering. We have also used the fact that on functions covariant derivative coincides with the ordinary one. Now, we will focus on solving (2.5). If we multiply both sides in (2.5) by $2g^{-1}$ and then take the trace we find that

$$tr[\sqrt{\Sigma}] = -\frac{4}{3}C_1 - \frac{1}{3}C_2\tilde{T}^\mu{}_\mu, \quad (2.10)$$

where we define

$$\tilde{T}^\mu{}_\mu \equiv tr[g^{-1}\tilde{T}] = g^{\mu\rho}\tilde{T}_{\rho\mu}. \quad (2.11)$$

When (2.10) is substituted in (2.5) one obtains

$$g\sqrt{\Sigma} + \sqrt{\Sigma}^T g = -\frac{2}{3}(C_1 + C_2\tilde{T}^\mu{}_\mu)g + 2C_2\tilde{T}. \quad (2.12)$$

Furthermore, by using the symmetry of the matrix $g\sqrt{\Sigma}$ [16] we can write (2.12) as

$$\sqrt{\Sigma} = -\frac{1}{3}(C_1 + C_2\tilde{T}^\mu{}_\mu)\mathbf{1}_4 + C_2g^{-1}\tilde{T}. \quad (2.13)$$

Here $\mathbf{1}_4$ is the four dimensional unit matrix. If we square both sides and isolate f on the left hand side we obtain

$$f = \mathcal{G}', \quad (2.14)$$

where we introduce $\mathcal{G}' = g\mathcal{G}^2$ with

$$\mathcal{G} = -\left(\frac{C_1 + C_2\tilde{T}^\mu{}_\mu}{3}\right)\mathbf{1}_4 + C_2g^{-1}\tilde{T}. \quad (2.15)$$

The constraint equation (2.14) together with the effective Einstein equations (2.6) are the remainder equations of the action (2.1) to be solved upon specifying the effective matter energy-momentum tensor \tilde{T} subject to the conservation constraint (2.8). We should remark that as we have not specified the form of the physical metric g , and the effective energymomentum tensor \tilde{T} (2.14) is in its most general matrix form which must be satisfied by $f, g, \{\phi^a\}$, and \tilde{T} .

3 Diagonal metric solutions

In this section we will derive the general solutions of (2.14) for a diagonal choice of the physical and the induced fiducial metrics. Firstly, let us state

that when g and f are assumed to be diagonal then the building blocks $\Sigma = g^{-1}f$ and $\sqrt{\Sigma}$ in (2.1) also become diagonal. This causes some degree of reduction of the non-linearity of the gravity sector in (2.1). Next, if we focus on our solutions we quickly observe via (2.14) and (2.15) that under the diagonality assumption of the metrics the effective energy-momentum tensor \tilde{T} must also be chosen diagonal for consistency. Now, in component form (2.14) can be written as

$$\partial_\mu \phi^a \partial_\nu \phi^b \bar{f}_{ab}(\phi^c) = \mathcal{G}'_{\mu\nu}. \quad (3.1)$$

Let us also assume the fiducial metric diagonal too, namely $\bar{f} = \text{diag}(f_{00}, f_{ii})$ where $i = 1, 2, 3$. In this case the equations to be satisfied in (3.1) take the following form

$$\sum_{a=0}^3 (\partial_\mu \phi^a)^2 f_{aa}(\phi^b) = \mathcal{G}'_{aa} \quad \forall \mu, \quad (3.2)$$

$$\sum_{a=0}^3 \partial_\mu \phi^a \partial_\nu \phi^a f_{aa}(\phi^b) = 0, \quad \text{when } \mu \neq \nu.$$

A straightforward solution to the second set of equations above can be obtained by setting

$$\partial_\mu \phi^a \partial_\nu \phi^a = 0, \quad \forall a, \quad \text{and } \mu \neq \nu. \quad (3.3)$$

Now, as also discussed in [15] one can multiply both sides of the first set of equations in (3.2) by $\partial_\alpha \phi^\alpha \partial_\beta \phi^\beta \partial_\gamma \phi^\gamma$ (where there is no sum on the indices, and with $\alpha, \beta, \gamma \neq \mu$ for all μ) then successive use of (3.3) in the result leads to the set of equations

$$(\partial_a \phi^a)^2 f_{aa} = \mathcal{G}'_{aa} \quad \forall a, \quad (3.4)$$

where there is no sum on a again. One can directly check that a particular set of solutions to (3.3) is obtained by setting

$$\partial_{\mu \neq a} \phi^a = 0, \quad \forall \mu, a. \quad (3.5)$$

Finally, now if we find a set of solutions for ϕ^a 's and f_{aa} 's which satisfy (3.4) and (3.5) for the diagonal choice of g, \bar{f}, \tilde{T} then they also satisfy the ansatz

constraint equation (2.14) so that they become solutions of (2.4) namely the Stückelberg sector of (2.1). Firstly, let us focus on Eq. (3.5) which states that for each $a = 0, 1, 2, 3$ the scalar field ϕ^a must be a function of x^a only. Thus, keeping this fact in mind if we choose the diagonal fiducial metric components as

$$f_{aa} = \frac{\mathcal{G}'_{aa}}{(F_a(x^a))^2}, \quad (3.6)$$

where for each $a = 0, 1, 2, 3$ we have introduced the arbitrary integrable functions F_a 's which only depend on the corresponding coordinate x^a then

$$\phi^a(x^a) = \pm \int F_a(x^a) dx^a, \quad (3.7)$$

become the solutions of (3.4). One can instantly realize that these scalar fields also satisfy (3.5) thus, (3.3). Therefore, they admit a family of solutions of the constraint equation (2.14) when the diagonal fiducial metric components are chosen to be (3.6). On the other hand for a diagonal choice of g and \tilde{T} we can explicitly construct the diagonal matrix components \mathcal{G}'_{aa} 's. Firstly, let us observe

$$\tilde{T}^\mu{}_\mu = \sum_{a=0}^3 \frac{\tilde{T}_{aa}}{g_{aa}}. \quad (3.8)$$

If we define

$$M = -\left(\frac{C_1 + C_2 \tilde{T}^\mu{}_\mu}{3}\right), \quad (3.9)$$

then we have

$$\mathcal{G}'_{aa} = M^2 g_{aa} + 2C_2 M \tilde{T}_{aa} + C_2^2 \frac{\tilde{T}_{aa}^2}{g_{aa}}, \quad (3.10)$$

where there is no sum on the index a on the right hand side. We should state once more that the off-diagonal elements of \mathcal{G}' are zero. As a result we have shown that for a diagonal metric solution of the modified Einstein equations (2.6) and a diagonal effective energy-momentum tensor source introduced in the ansatz (2.5) when (3.6) is determined via the substitution of (3.10) then (3.7) become the solutions of the Stückelberg sector of (2.1).

4 Cosmological solutions

In this section we will consider the FLRW metric ansatz for the modified Einstein equations (2.6). Thus, as a special case of the set of solutions

constructed in the previous section we will take the physical metric as the FLRW metric in spherical coordinates

$$g = -dt^2 + \frac{a^2(t)}{1 - kr^2} dr^2 + a^2(t)r^2 d\theta^2 + a^2(t)r^2 \sin^2\theta d\varphi^2, \quad (4.1)$$

with k being the scalar curvature of the 3-space and $a(t)$ being the scale factor. Like the physical ideal fluid whose energy-momentum tensor can be written as $T = \text{diag}(\rho, p, p, p)$ in the momentarily co-moving frame, for obtaining the homogeneous and isotropic metric solution (4.1) we also take the effective source of our ansatz as a perfect fluid. Therefore, for the co-ordinate frame $\{x^\mu\}$ in which the metric components can be read via (4.1) as

$$g_{00} = -1, \quad g_{11} = \frac{a^2}{1 - kr^2}, \quad g_{22} = a^2 r^2, \quad g_{33} = a^2 r^2 \sin^2\theta, \quad (4.2)$$

we can set the effective ideal fluid energy-momentum tensor from its general definition as

$$\tilde{T}_{\mu\nu} = (\tilde{\rho}(t) + \tilde{p}(t))U_\mu U_\nu + \tilde{p}(t)g_{\mu\nu}. \quad (4.3)$$

Thus, now if we take $U_0 = 1$, and $U_i = 0$ for $i = 1, 2, 3$ in the co-ordinate frame $\{x^\mu\}$ we have

$$\tilde{T} = \text{diag}(\tilde{\rho}(t), \tilde{p}(t)g_{11}, \tilde{p}(t)g_{22}, \tilde{p}(t)g_{33}), \quad (4.4)$$

with $\tilde{\rho}$ being the effective energy density, and \tilde{p} the effective pressure built out of the Stückelberg scalars which have an effective contribution to the metric equation via the non-physical source (4.4).

The Stückelberg sector:

For the FLRW metric choice (4.1) the contraction $\tilde{T}^\mu{}_\mu = g^{\mu\nu}\tilde{T}_{\mu\nu}$ of (4.4) becomes

$$\tilde{T}^\mu{}_\mu = 3\tilde{p} - \tilde{\rho}. \quad (4.5)$$

The components of \mathcal{G}' which enter into the definition of the fiducial metric (3.6) that give rise to the Stückelberg field solutions (3.7) can be explicitly given as

$$\mathcal{G}'_{00} = -M^2 + 2C_2 M \tilde{\rho} - C_2^2 \tilde{\rho}^2, \quad (4.6)$$

$$\mathcal{G}'_{ii} = (M^2 + 2C_2 M \tilde{p} + C_2^2 \tilde{p}^2)g_{ii},$$

where $i = 1, 2, 3$ again. When the nature of the effective ideal fluid composing \tilde{T} is arbitrarily and independently specified in addition to the physical ideal fluid, the scale curvature k is fixed and the scale factor $a(t)$ is solved from the modified Einstein equations (2.6) for the metric ansatz (4.1) these functions explicitly determine the fiducial metric (3.6) for which the scalar fields (3.7) form up the solutions of the Stückelberg sector together with the metric solution (4.1) of the metric sector which we have dynamically decoupled from the scalars.

The FLRW Dynamics:

Now, if we use the FLRW metric (4.1) in the on-shell metric equation (2.6) which is in Einstein form up to modification by an effective source of energy-momentum tensor which in our case we take to be a perfect fluid like the physical matter then we get the first Friedmann equation

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{G_N}{3}\rho - \frac{C_2 m^2}{3}\tilde{\rho} - \frac{k}{a^2} - \frac{\tilde{\Lambda}}{3}, \quad (4.7)$$

and via the first and the second Friedmann equations the acceleration equation

$$\frac{\ddot{a}}{a} = - \left[\frac{G_N(\rho + 3p) - C_2 m^2(\tilde{\rho} + 3\tilde{p})}{6} \right] - \frac{\tilde{\Lambda}}{3}. \quad (4.8)$$

When the cosmological metric ansatz (4.1) is used in the physical perfect fluid matter energy-momentum conservation equation $\nabla^\mu T_{\mu\nu} = 0$, and also in the similar constraint equation (2.8) for the effective perfect fluid (4.4) we obtain the continuity or the fluid equations

$$\dot{\rho} + 3H(\rho + p) = 0 \quad , \quad \dot{\tilde{\rho}} + 3H(\tilde{\rho} + \tilde{p}) = 0, \quad (4.9)$$

respectively. Therefore, when the equation of states are determined for the physical and the effective ideal fluids from (4.7), (4.8), (4.9) one can solve the scale factor $a(t)$ and the energy densities $\rho(t)$ and $\tilde{\rho}(t)$. Then one can use these solutions in (4.5) and (4.6) to explicitly obtain the building blocks of the fiducial metric (3.6). After specifying the arbitrary integrable functions $\{F_a(x^a)\}$ one entirely determines the background metric, and finds the Stückelberg scalar solutions from (3.7). As a result one obtains the cosmological FLRW metric, and the Stückelberg scalar field solutions of (2.1) for

the specially constructed fiducial metric. We have to stress that these solutions work for any scale curvature without putting restrictions on it. Before concluding we have to make an important remark regarding our solution generation method; the reader should observe that although the equation of state of the matter ideal fluid is determined by its physical nature the effective ideal fluid which emerges as a mathematical source of our solution ansatz (2.5) has no restrictions on it apart from (2.8) which leads to the second equation in (4.9). For example the physical perfect fluids have the equation of state in the form $p = w\rho$, however, our construction suggests no special form for the equation of state for the effective ideal fluid, namely it seems that one can assign any form to it $\tilde{p} = f(\tilde{\rho})$.

5 Conclusion

After decoupling completely the metric sector of the minimal massive gravity from the Stückelberg scalar fields of the mass term by introducing an ansatz we have focussed on the solutions satisfying the ansatz constraint on the background metric and the Stückelberg fields. We have shown that in this on-shell formalism the metric field equations reduce to the Einstein equations modified solely by a contribution of an effective energy-momentum tensor which appears as a source in the ansatz we have proposed. We have later constructed a formal family of solutions for the background fiducial metric and the Stückelberg fields in the presence of a diagonal physical metric. In the last section we have considered the cosmological applications within this scheme. By assuming the homogeneous and isotropic FLRW metric in the modified Einstein equations where the scalars do not appear explicitly we have given the corresponding solutions of the scalars and the background metric which satisfy the ansatz constraint and lead to the above-mentioned decoupling. Due to the necessity of homogeneity and the isotropy of the FLRW cosmology we have restricted ourselves to an ideal fluid form for the effective matter. Then via the FLRW metric used in the modified Einstein sector we have also derived the cosmological equations for the scale factor dynamics, namely the first Friedmann and the acceleration equations.

The method of solving the minimal massive gravity field equations that is constructed in this work diverges from the one widely used in the literature. Instead of pre-determining the background metric from the start we find it as a solution of an ansatz which enables the decoupling of the scalars from the

metric equation. Thus, we design the necessary fiducial metric to generate solutions to the theory in this sense. We have performed this analysis first for a more general diagonal physical metric then as a special case for the FLRW cosmological one. In this manner, after determining the background fiducial metric we also obtain the scalar field solutions when the physical metric is specified. We should state that apart from constructing explicit solutions this kind of formalism may have essential physical consequences. The effective matter energy-momentum tensor which appears as an overall mathematical source within the solution constraint is entering in the cosmological equations as completely an arbitrary effective ideal fluid whose equation of state can be chosen in any form to generate various solutions. This may be interesting in two aspects; firstly, one can design the necessary forms of effective matter (design the corresponding solution instead) to cure the dark energy or the dark matter problems in the presence of accompanying physical matter, secondly, one can work out various exotic solutions arising from this new kind of gravitational matter. If on the other hand, one considers solely the self-acceleration issue we observe that although in our solution scheme the modified on-shell metric equation has an effective cosmological constant the new coming terms are suppressed by the graviton mass. However, it is also obvious that having an effective ideal fluid source degree of freedom in the cosmological equations guaranties the existence of self-accelerating solutions in the absence of the cosmological constant. The solution construction formalism proposed in this work can also be extended to cover the more general massive gravity case whose Lagrangian has more non-linearly involved mass terms. In this manner, one can also derive, and study new cosmological solutions of the general massive gravity. We leave this discussion here to appear in a separate work [17]. However, in this direction we must also stress on the problems arising for the known types of cosmological solutions of massive gravity [18, 19], as well as the problems that are reported via the inspection of the theory from a more general perspective [20].

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